Analytic Guidance and Control of Low-Thrust Rendezvous

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The controls for rendezvous with minimum fuel expenditure and optimal guidance corrections for disturbances are studied for a low thrust vehicle in Martian orbit. The target orbit, in general form, is simplified for a circular target orbit. Analytic suboptimal controls are found which yield good performance with respect to a physical cost functional but have the basic computational advantages of small computer storage and no numerical matrix inversions. Standard second variation methods for the guidance corrections cannot be used with suboptimal controls as references. A method which yields a modified cost functional by solving an inverse problem in the calculus of variations is presented. The suboptimal controls become optimal for the modified functional. The second variation is then applied to the modified cost functional to yield the guidance corrections.

Nomenclature

a_r = specific thrust acceleration—radial
a_θ = specific thrust acceleration—transverse
A_i = prespecified weighting factors ($i = 1,3$)
C_i = weighting factors in integrand L_2 ($i = 1,12$)
f = $[f_i]$ = system dynamics vector ($i = 1,4$)
G = $G(x,k_f)$ = quadratic terminal term in J_1 and J_2
H_1,H_2 = Hamiltonians
J_1,J_2 = cost functionals
k = $[r^2d\theta/dt]^2$ = specific angular momentum squared
L_1,L_2 = integrands of cost functionals
m = $[m_i] = [a_\theta,a_r]$ = control vector
Q_i = feedforward guidance correction controls
r = radial distance from Mars center
t = time
u = $1/r$ = dummy variable
v_r = radial velocity
v_θ = transverse velocity
x = $[x_i] = [\theta,r,v_r,t]$ = state vector
α = weighting factor
β = small constant
δm = $[\delta m_i] = [\delta a_\theta,\delta a_r]$ = perturbed controls
δx = $[\delta x_i] = [\delta\theta,\delta r,\delta v_r,\delta t]$ = perturbed states
$\delta\lambda$ = $[\delta\lambda_i] = [\delta\lambda_1,\delta\lambda_2,\delta\lambda_3,\delta\lambda_4]$ = perturbed multipliers
η = small constant
θ = angular distance from reference axis
$\dot\theta_T$ = angular velocity of circular target—constant
λ = $[\lambda_i] = [\lambda_1,\lambda_2,\lambda_3,\lambda_4]$ = Lagrange multiplier vector
μ_m = Martian gravitational constant
μ = $\mu_m/[1 + (\beta^2 + \eta\beta)/2]$
ρ_1,ρ_2 = solutions for control vectors
ψ = control angle
ω = $[1 + \eta\beta/2 - \beta^2/16]^{1/2}/\beta$ = const

Subscripts

f = final value of quantity
N = normalizing quantity
0 = initial value of quantity
T = target quantity
1 = relates to physical cost functional
2 = relates to modified cost functional

Superscripts

* = optimal quantity
^ = reference trajectory
T = transpose

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I. Introduction

The paper determines analytic controls for rendezvous and guidance corrections for disturbances to be used for a low-thrust space vehicle, initially in a specified Martian orbit, to rendezvous at a higher terminal orbit.[1]

The optimal controls and guidance for rendezvous can be determined "numerically" by solving the well-known two-point boundary problem and the associated matrix Ricatti equation for the linear optimal feedback gains in the second variation method.[2–4] These numerical solutions result in the minimum fuel cost of control and guidance. But penalties in computer storage facilities and computer time are also important. In addition, the small computer onboard the Martian vehicle will be hard pressed to meet the large storage requirements of the numerical trajectories used in the second-variation method. This problem, coupled with the difficulties of iteration types of numerical techniques for determining optimal controls, including their associated matrix inversions, make feasible analytic solutions higly desirable. Later sections will develop analytic solutions for the controls with favorable fuel costs compared to the numerical optimal. The method of treating the guidance problem for disturbances, in Sec. VI, results in negligible storage requirements for the calculation of the feedback gains. In addition, no numerical matrix inversions are required.

The paper makes use of parens, ( ), to indicate a functional relationship for variables. For example, $x(t)$ means that the variable x is a function of the variable t.

II. Equations of Motion—"k" Domain

The motion of a continuous, low-thrust vehicle in an assumed inverse-square Martian gravitational field is shown in Fig. 1. The familiar equations of motion with time t as the independent variable are

$$d\theta/dt = v_\theta/r, \; dr/dt = v_r$$
$$dv_r/dt = [v_\theta^2/r - \mu_m/r^2] + a_r \qquad (1)$$
$$dv_\theta/dt = -v_r v_\theta/r + a_\theta$$

Now, using the square of the specific angular momentum $k = [r^2 d\theta/dt]^2 = [rv_\theta]^2$, rather than time as the independent variable, yields the equations of motion[5,6]:

$$\begin{aligned} f_1 &= d\theta/dk = 1/[2r^3 a_\theta] \\ f_2 &= dr/dk = v_r/2k^{1/2} r a_\theta \\ f_3 &= dv_r/dk = [\tfrac{1}{2}k^{1/2}a_\theta]\{k/r^4 - \mu_m/r^3 + a_r/r\} \\ f_4 &= dt/dk = 1/[2k^{1/2} r a_\theta] \end{aligned} \qquad (2)$$

Note that the equation for v_θ has been eliminated since k, r, and v_θ are not equal to zero. These formulations, though formidable looking, will be shown to yield simple analytical solutions.

III. Evolution of Problem Method of Approach to Guidance

Let us consider first the optimal rendezvous problem. Starting from a specified Martian orbit it is desired to determine the controls for rendezvous at a terminal orbit while minimizing fuel in the low-thrust vehicle.

The equivalent mathematical formulation of this type of optimal problem requires the minimization of a cost functional J_1 in the time domain of the form:

$$J_1 = G(x,t_f) + \int_{t_0}^{t_f} [\tilde{L}_1(x,m,t)]dt \qquad (3)$$

where $\tilde{L}_1$ is the normalized thrust acceleration:

$$\tilde{L}_1 = [a_\theta^2(t) + a_r^2(t)]/a_N^2 \qquad (4)$$

which is equivalent to minimizing fuel consumption.[7] Here a_N is the magnitude of a normalized thrust acceleration.

In order to simplify the discussion, hereafter the terminal orbit is considered circular. Four independent quantities must be matched at the open terminal time t_f for rendezvous to occur. These quantities could be the angular position (θ), the velocities in the transverse (v_θ) and radial (v_r) directions, and the radial distance from the planet's center (r).

Alternatively, the square of the specific angular momentum k may be matched at the terminal orbit in place of the transverse velocity v_θ. Here

$$k(t) = [r^2 d\theta/dt]^2 = [r(t)v_\theta(t)]^2 \qquad (5)$$

Now suppose in addition to changing a term, the problem is transformed from the time domain to the domain where k is the independent variable. There is now no need for the new term constraining the value of $k(t_f)$. The independent variable k now varies from its given value on the initial trajectory k_0 to the specified value of the unthrusted target trajectory k_T, i.e.,

$$k(t_f) = k_f = k_T \qquad (6)$$

Additionally, a great computational advantage is obtained by working in the k domain. The problem is transformed from an open terminal time t_f to a specified value of the independent variable k_f given in Eq. (6). The cost functional J_1 to be minimized now becomes

$$J_1 = G(x,k_f) + \int_{k_0}^{k_f} \{\tilde{L}_1(x,m,k)\}(dt/dk)dk \qquad (7)$$

with $\tilde{L}_1$ of Eq. (4) in the k domain

$$\tilde{L}_1 = [a_\theta^2(k) + a_r^2(k)]/a_N^2 \qquad (8)$$

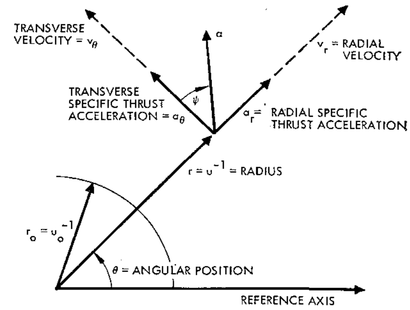

Fig. 1 Coordinate system.

These considerations then yield a terminal term G in the quadratic form:

$$G(x,k_f) = A_1\{\theta(k_f) - \theta_T(k_f)/\theta_N\}^2 + A_2\{r(k_f) - r_T(k_f)/r_N\}^2 + A_3\{v_r(k_f) - v_{rT}(k_f)/v_N\}^2 \qquad (9)$$

The θ_N, r_N, and v_N are normalizing factors. The θ_T, r_T, and v_{rT} are the angular position, radial distance, and radial velocity of the unthrusted circular target orbiter. These quantities are

$$\theta_T(k) = \theta_T(t_0) + [t(k) - t_0]\dot{\theta}_T, \; \theta_T(t_0) = 0$$
$$r_T(k) = \text{const}, \; v_{rT}(k) = \text{const} = 0 \qquad (10)$$

The $\dot{\theta}_T$ is the angular velocity of the circular target orbiter and is constant.

The terminal term G thus matches three terminal value quantities of the moving continuous thrust vehicle and the unthrusted circular target orbiter for rendezvous. In the two-dimensional problem a fourth independent quantity, needed to insure a rendezvous at the proper time, is the final value of k_f, i.e., $k_f = k_T$ of the target (k_T = const). It is again noted that the terms involving the transverse velocity v_θ are eliminated from the discussion when k is used as the independent variable. This elimination is caused by the fact that k, r, and v_θ are not independent.

The A_1, A_2, and A_3 are appropriate weighting factors and are prespecified quantities. The mathematical statement of the optimal problem becomes to find the controls a_θ^* and a_r^* that minimize J_1 of Eqs. (7–9) subject to the dynamics of Eq. (2) and the specified initial conditions.

The solutions a_θ^* and a_r^* to this fuel optimal rendezvous problem must be obtained numerically. To circumvent the numerical difficulties mentioned in Sec. I, this paper departs from the more traditional forms of guidance with the following assertion, discussed in Sec. V.

A small modification ΔJ_1 of the fuel optimal cost functional in Eqs. (7–9), will yield analytic solutions when the $J_1 + \Delta J_1$ is minimized. The word "small" refers not only to the numerical size of ΔJ_1, but also to its influence upon the final solution. Depending upon the form of ΔJ_1 the solutions obtained will tend to minimize fuel while completing a rendezvous and thus will be close to fuel optimal. With these comments and the assertion in mind, the problem is modified as follows. Find $J_2 = J_1 + \Delta J_1$ which when minimized subject to Eq. (2) yields analytic controls for rendezvous near the fuel optimal. Guidance corrections for disturbances are obtained from J_2 using the analytic controls and trajectories as references in a quasi-optimum perturbation scheme (Sec. VI).

IV. Reference Trajectory and Controls

The transformation of the equations of motion from the time domain into the k domain yielded Eq. (2). These equations, at first, appear even more formidable than the usual form. But consider the controls:

$$a_\theta = \tfrac{1}{2}\beta k/r^3; \; a_r = \tfrac{1}{2}\eta\beta k/r^3 \qquad (11)$$

where η and β are constants (small for low thrust).

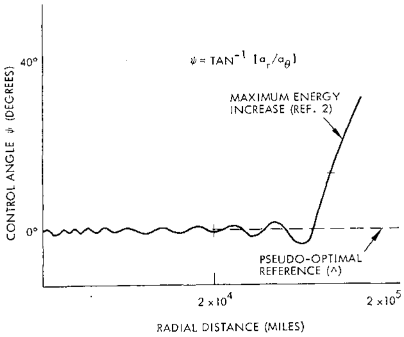

Fig. 2 Comparison of control angles.

Before substituting these controls into Eq. (2), consider, temporarily, an additional change of dependent variables: Let $u = 1/r$. Then with the definitions of $v_r = dr/dt = (dr/dk)(dk/dt)$, variable u, and controls in Eq. (11), the f_2 and f_3 equations of Eq. (2) may be simplified to yield:

$$d^2u/dk^2 + 3/2k\,du/dk + \{(1 + \eta\beta/2)/\beta^2k^2\}u = \mu_m/\beta^2k^3 \quad (12)$$

Equation (12) is recognized as an Euler type differential equation with a regular singular point. The solution, with initial conditions at k_0, is

$$u(k) = 1/r(k) = \mu/k + [k/k_0]^{-1/4}([1/r_0 - \mu/k_0]\cos\{\omega\ln[k/k_0]\} + \{1/\omega\}[3\mu/4k_0 + 1/4r_0 - v_{r_0}/\beta k_0^{1/2}]\sin[\omega\ln\{k/k_0\}]) \quad (13)$$

where μ/k is the particular solution to Eq. (12) and

$$\mu = \mu_m/[1 + (\beta^2 + \eta\beta)/2] \quad (14a)$$

$$\omega = [1 + \eta\beta/2 - \beta^2/16]^{1/2}/\beta \quad (14b)$$

The restriction on the solution that $1 + \eta\beta/2 - \beta^2/16 \geq 0$ holds for low thrust, i.e., small values of η and β.

Suppose initial conditions are picked as

$$\theta(k_0) = \theta_0;\ r(k_0) = r_0 = k_0/\mu$$
$$v_r(k_0) = v_{r_0} = \beta\mu/k_0^{1/2};\ t(k_0) = t_0 \quad (15)$$

The resultant extremely simple trajectory from Eq. (13) and the solution to f_1 and f_4 of Eq. (2) is

$$\hat{\theta}(k) = \theta_0 + 1/\beta\ln[k/k_0];\ \hat{r}(k) = k/\mu$$
$$\hat{v}_r(k) = \beta\mu/k^{1/2};\ \hat{t}(k) = t_0 + [k^{3/2} - k_0^{3/2}]/1.5\beta\mu^2 \quad (16)$$

The hat (^) refers to the reference trajectory. These trajectories are the equations of a logarithmic spiral. Since the control angle ψ is

$$\psi = \arctan[\hat{a}_r/\hat{a}_\theta] \approx \eta \quad (17)$$

In the continuous low-thrust problem, η is small and the thrust is mainly in the transverse direction (Fig. 1), which is very close to fuel optimum[8] (Fig. 2). The controls in Eq. (11), though not optimal, will be shown to represent a good choice for rendezvous with small fuel expenditure.

V. Modified Functional J_2

A. Modified Problem Formulation

The analytic controls $\hat{a}_\theta$ and $\hat{a}_r$ in Eq. (11) can be very useful. Since they are close to fuel optimal it seems logical that a small modification of the fuel optimal cost functional J_1, might yield a modified functional J_2 which has Eq. (11) as optimal controls. In other words, given a J_1, the controls $\hat{a}_\theta$ and $\hat{a}_r$ will not be optimal. We then try to find a J_2 for which the controls $\hat{a}_\theta$ and $\hat{a}_r$ are optimum. Problems of this type are called inverse problems in the calculus of variation.[9]

The motivation for finding a J_2 is to eliminate the numerical storage problems involved in the second-variation method when developing a guidance scheme. But there exists an implicit restriction on the J_2. Cost functional J_1 represents a physical situation. The functional J_2 represents a mathematical situation. Hence, the modified functional J_2 must have physical significance when compared to J_1. The difference between them must be small as discussed previously in Sec. III.

Assume a modified cost functional J_2 in the quadratic and reciprocally quadratic form

$$J_2 = G(x,k_f) + \int_{k_0}^{k_f}[L_2(x,m,k)]dk \quad (18)$$

The terminal term G is given by

$$G(x,k_f) = A_1\left[\frac{\theta(k_f) - [t(k_f) - t_0]\dot{\theta}_T}{\theta_N}\right]^2 + A_2\left[\frac{r(k_f) - r_T}{r_N}\right]^2 + A_3\left[\frac{v_r(k_f)}{v_N}\right]^2 \quad (19)$$

The integral L_2 is

$$L_2 = [C_1a_\theta^2 + C_2a_r^2] + [C_3/a_\theta^2 + C_4/a_r^2] + [C_5\theta^2 + C_{6}r^2 + C_7v_r^2 + C_8t^2] + [C_9/\theta^2 + C_{10}/r^2 + C_{11}/v_r^2 + C_{12}/t^2] \quad (20)$$

The quantities $\theta, r, v_r, t, a_\theta$, and a_r are not equal to zero unless the corresponding C_i is also equal to zero. The form of L_2 also assumes the weighting factor dt/dk of $\tilde{L}_1$ in Eq. (7) has been evaluated along the reference trajectory. The J_1 for comparison purposes becomes

$$J_1 = G(x,k_f) + \int_{k_0}^{k_f}[L_1(x,m,k)]dk \quad (21)$$

The terminal term G is the same as for J_2 and is given by Eq. (19), whereas the integrand is

$$L_1(x,m,k) = [\tilde{L}_1]dt/dk = [k^{1/2}/\beta\mu^2a_N^2][a_\theta^2(k) + a_r^2(k)] \quad (22)$$

The inverse problem for the modified function J_2 is to determine the C_i, which may be functions of k, that yield the analytic solutions of Eqs. (11) and (16) as the minimum of J_2. The C_i are subject to certain restrictions. For a minimum of J_2 look for $C_i \geq 0$. Also, it is hoped that any additional terms in J_2, which differ from J_1, will be zero or small. In order to obtain the C_i it is necessary to solve for the Lagrange multipliers in the Euler-Lagrange equations of J_2.

Thus, when J_2 of Eqs. (18–20) is minimized subject to the dynamics of Eq. (2) with initial conditions as in Eq. (15), find the C_i such that the resultant controls are

$$\hat{a}_\theta = \beta k/2r^3 = \beta\mu^3/2k^2,\ \hat{a}_r = \eta\beta k/2r^3 = \eta\beta\mu^3/2k^2 \quad (23)$$

with corresponding reference trajectory of Eq. (16).

In the remainder of the text variables with a hat (^) denote evaluations along the reference controls and trajectory of Eqs. (11) and (16).

B. Lagrange Multipliers and the Reference Trajectory

The Hamiltonian H_2 for J_2 is

$$H_2 = L_2 + \sum_{i=1}^{4}\lambda_if_i \quad (24)$$

where L_2 is the integrand of Eq. (20) and the λ_i are the Lagrange multipliers that adjoin the dynamics f_i of Eq. (2). The Euler-Lagrange equations for minimum J_2 become

$$d\hat{x}_i/dk = f(\hat{x},\hat{m},k) = [\hat{f}i] \quad (25)$$

$$d\hat{\lambda}/dk = -(\partial H_2/\partial x)(\hat{x},\hat{m},\hat{\lambda},k) = -[(\partial \hat{L}_2/\partial x) + \hat{\lambda}^T \partial \hat{f}/\partial x] \quad (26)$$

$$0 = (\partial H_2/\partial m)(\hat{x},\hat{m},\hat{\lambda},k) = \partial \hat{L}_2/\partial m + \hat{\lambda}^T \partial \hat{f}/\partial m \quad (27)$$

with boundary conditions of Eq. (15) and transversality condition[10]

$$\hat{\lambda}(k_f) = [\hat{\lambda}_i(k_f)] = (\partial G/\partial x)(\hat{x},k_f) \quad (28)$$

The reference trajectory ($\hat{\ }$) of Eqs. (11) and (16) is required to satisfy Eqs. (25–28).

C. Reference Trajectory—Solution for Lagrange Multipliers and C_i

Equations (25–28) together with the physical cost functional J_1 of Eqs. (21–22) are now used to determine the C_i.

1. *The quantities C_5, C_8, C_9, C_{12}*

The assumption is now made that for the initial pseudo-optimal trajectory the final angle is not constrained, because adjustment of the reference trajectory value of θ_0, the initial angle, can yield any desired value of the final angle.[2] It is important to note that here we are concerned with the reference trajectory and not the problem of guidance corrections for disturbances. When predetermining a reference trajectory, any value of θ_0 may be chosen. Thereafter θ_0 is a fixed value and the angular constraint along a perturbed path due to disturbances must exist. Thus, for the reference trajectory only, without any loss in generality, $\lambda_1(k)$ may be taken as equal to zero for all k. Then the Euler-Lagrange equation of Eq. (26) for λ_1 is

$$d\lambda_1/dk = \partial L_2/\partial\theta = -[2C_5\hat{\theta} - 2C_9/\hat{\theta}^3] = 0 \quad (29)$$

which implies

$$C_5 = C_9 = 0 \quad (30)$$

In a similar manner, leaving the final time unconstrained so $\lambda_4(k) = 0$ for all k yields

$$C_8 = C_{12} = 0 \quad (31)$$

2. *The quantities C_6, C_7, C_{10}, C_{11}*

Substituting the reference trajectory ( ) into the adjoint equations of Eq. (26) yields the differential equations to be satisfied for $\hat{\lambda}_2$ and $\hat{\lambda}_3$ [noting μ in Eq. (19a) and $\lambda_1 = \lambda_4 = 0$ as previously]:

$$d\hat{\lambda}_2/dk = -\partial \hat{L}_2/\partial r + (1/k)\hat{\lambda}_2 + (\mu^2/\beta k^{5/2})[1 - \eta\beta - 3\beta^2/2]\hat{\lambda}_3 \quad (32)$$

$$d\hat{\lambda}_3/dk = -\partial \hat{L}_2/\partial v_r - (k^{1/2}/\beta\mu^2)\hat{\lambda}_2 \quad (33)$$

Eliminating $\hat{\lambda}_3$ between these two equations yields

$$\frac{d^2\hat{\lambda}_2}{dk^2} + \frac{3}{2k}\frac{d\hat{\lambda}_2}{dk} + \frac{(1-\eta\beta-3\beta^2)}{\beta^2k^2}\hat{\lambda}_2 = \frac{\mu^2}{\beta k^{5/2}}\left(1-\eta\beta-\frac{3\beta^2}{2}\right)\left[\frac{\partial \hat{L}_2}{\partial v_r}\right] - \frac{5}{2k}\left[\frac{\partial \hat{L}_2}{\partial r}\right] - \frac{d}{dk}\left[\frac{\partial \hat{L}_2}{\partial r}\right] \quad (34)$$

As in Sec. IV, this differential equation is of the Euler type, with a regular singular point. Recalling the method used previously to solve for the reference trajectory, suppose λ_2 was identically equal to its particular solution. The particular solution must identically satisfy the boundary or transversality condition of Eq. (28):

$$\hat{\lambda}_2(k_f) = 2A_2[\hat{r}(k_f) - r_T]/r_N^2 = (A_2/r_N^2)(k_f/\mu_m)[\beta^2 + \eta\beta] \quad (35)$$

The solution for $\hat{\lambda}_2$ will be tried in the form:

$$\hat{\lambda}_2(k) = (A_2/r_N^2)(k/\mu_m)[\beta^2 + \eta\beta] \quad (36)$$

Similarly, the solution for $\hat{\lambda}_3$ will be tried in the form of its transversality condition as:

$$\hat{\lambda}_3(k) = 2A_3v_r(k)/v_N^2 = (A_3/v_N^2)2\beta\mu/k^{1/2} \quad (37)$$

Substitution of these solutions for $\hat{\lambda}_2$ and $\hat{\lambda}_3$ into Eq. (32), with $\partial \hat{L}_2/\partial r = 2C_6\hat{r} - 2C_{10}/r^3$ yields

$$\frac{d\hat{\lambda}_2}{dk} = \frac{A_2}{r_N^2}\frac{[\beta^2+\eta\beta]}{\mu_m} = -\left[2C_6\hat{r} - \frac{2C_{10}}{\hat{r}^3}\right] + \frac{A_2}{r_N^2}\frac{[\beta^2+\eta\beta]}{\mu_m} + \frac{A_3}{v_N^2}\frac{2\mu^3}{k^3}\left[1-\eta\beta-\frac{3\beta^2}{2}\right] \quad (38)$$

Using $\hat{r} = k/\mu$, and noting that all C_i must satisfy $C_i \geq 0$ with $C_i = 0$ if possible, yields a solution

$$C_6 = A_3\mu^4(1-\eta\beta-3\beta^2/2)/v_N^2k^4,\ C_{10} = 0 \quad (39)$$

A similar type of equation, for C_7 and C_{11}, is derived from Eq. (33).

3. *The quantities C_1, C_2, C_3, C_4*

The remaining C_i are obtained by looking at the control gradient equations in Eq. (27) for $\hat{a}_r$ and $\hat{a}_\theta$, along with the solutions for the λ_i. For example, from $\partial \hat{H}_2/\partial a_r = 0$

$$2C_2\hat{a}_r - 2C_4/\hat{a}_r^3 = -[1/2k^{1/2}\hat{r}\hat{a}_\theta]\hat{\lambda}_3 = -2A_3/\mu v_N^2 \quad (40)$$

Comparing terms with the physical cost functional given in Eqs. (21–22) let

$$C_2 = \tfrac{1}{2}[k^{1/2}/\beta\mu^2a_N^2] = \alpha$$

Then, using $\hat{a}_r = \eta\beta\mu^3/2k^2$ yields

$$C_4 = (\eta^4\beta^4\mu^{12}/16k^8)[\alpha + (A_3/v_N^2)2k^2/\eta\beta\mu^4] \quad (41)$$

Similar types of equations may be derived for C_1 and C_3 from $\partial \hat{H}_2/\partial a_\theta = 0$. The solution to the inverse problem for J_2 is then complete.

D. Physical Interpretation of the Modified Functional J_2

Combining the C_i from the aforementioned three parts yields the modified functional

$$J_2 = G(x,k_f) + \int_{k_0}^{k_f} [L_2(x,m,k)]dk \quad (42)$$

where G is given in Eq. (19) and

$$L_2 = \alpha a_\theta^2 + \frac{\beta^4\mu^{12}}{16k^8} \times \left[\alpha - \frac{A_2}{r_N^2}\frac{2k^5}{\beta^2\mu^7\mu_m}[\beta^2+\eta\beta] + \frac{A_3}{v_N^2}\frac{2k^2}{\mu^4}\right]\frac{1}{a_\theta^2} + \alpha a_r^2 + \frac{\eta^4\beta^4\mu^{12}}{16k^8}\left[\alpha + \frac{A_3}{v_N^2}\frac{2k^2}{\eta\beta\mu^4}\right]\frac{1}{a_r^2} + \left[\frac{A_3}{v_N^2}\frac{\mu^4}{k^4}\left(1-\eta\beta-\frac{3\beta^2}{2}\right)\right]r^2 + \left[\frac{A_3}{v_N^2}\frac{1}{2k}\right]v_r^2 + \left[\frac{A_2}{r_N^2}\frac{\beta^2\mu[\beta^2+\eta\beta]}{2\mu_m}\right]\frac{1}{v_r^2} \quad (43)$$

where

$$\alpha = \tfrac{1}{2}[k^{1/2}/\beta\mu^2a_N^2] \quad (44)$$

Consider a low-thrust problem, starting from an orbit of 1000 miles above the Martian surface (r_0 = 3000 miles) to rendezvous with a circular target orbit at 2000 miles altitude

Table 1 Comparison of functionals

Variables		Coefficient C_1		J_2 terms		J_1 terms
Name	Value at k_0	Name	Value at k_0	Term	Value at k_0 (dimensionless)	Value at k_0 (dimensionless)
a_θ	1.6667×10^{-7} miles/sec	C_1	3.29×10^{12}	$C_1 a_\theta^2$	0.915×10^{-1}	1.830×10^{-1}
		C_3	2.54×10^{-15}	C_3/a_θ^2	0.915×10^{-1}	0.0
θ	0.0 rad	C_5	0.0	$C_5\theta^2$	0.0	0.0
		C_9	0.0	C_9/θ^2	0.0	0.0
r	3000 miles	C_6	1.235×10^{-10}	$C_6 r^2$	1.111×10^{-3}	0.0
		C_{10}	0.0	C_{10}/r^2	0.0	0.0
v_r	0.548×10^{-3} miles/sec	C_7	1.6667×10^{-4}	$C_7 v_r^2$	0.50×10^{-10}	0.0
		C_{11}	0.810×10^{-10}	C_{11}/v_r^2	0.27×10^{-3}	0.0
t	0.0 sec	C_8	0.0	$C_8 t^2$	0.0	0.0
		C_{12}	0.0	C_{12}/t^2	0.0	0.0

(r_T = 4000 miles). The values of the parameters are

$$\mu_m = 1.0 \times 10^4 \text{ miles}^3/\text{sec}^2;\ k_0 = 3.0 \times 10^7$$
$$k_f = 4.0 \times 10^7 \text{ miles}^4/\text{sec}^2;\ \eta = \beta = 3.0 \times 10^{-4}$$
$$A_1 = A_2 = A_3 = 1.0;\ r_N = 1.0 \text{ miles} \quad (45)$$
$$v_N = 0.01 \text{ miles/sec};$$
$$a_N = \beta\mu^3/2k_0^2 = 0.16667 \times 10^{-6} \text{ miles/sec}^2$$

Based upon these parameters, the values of the reference variables at k_0 are given in Table 1.

Now, separate the terms in J_2 into two groups; 1) Group 1 consists of $C_5\theta^2 + C_6r^2 + C_7v_r^2 + C_8t^2 + C_9/\theta^2 + C_{10}/r^2 + C_{11}/v_r^2 + C_{12}/t^2$. Table 1 lists the weighting coefficients C_i, and their approximate orders at k_0 after neglecting higher-order terms when compared to the $C_1a_\theta^2$ term. Only the C_6 term appears to have significant weighting when compared to the C_1 term. But along the reference trajectory at k_0, the ratio $C_6r^2/C_1a_\theta^2 \approx 1.2 \times 10^{-2}$, i.e., the C_6 term is about 1% of the C_1 term. Thus, the term in group 1 are physically very small along the reference trajectory.

2) Group 2 consists of $C_1a_\theta^2 + C_2a_r^2 + C_3/a_\theta^2 + C_4/a_r^2$. First note that substituting the numerical values of Eq. (45) into the C_3 and C_4 terms, using the reference trajectory and controls, yields

$$C_1 = C_2 = \alpha;\ C_3/\hat{a}_\theta^4 \approx \alpha;\ C_4/\hat{a}_r^4 \approx \alpha \quad (46)$$

Thus, rearranging the terms of group 2

$$[C_1\hat{a}_\theta^2 + C_3/\hat{a}_\theta^2] + [C_2\hat{a}_r^2 + C_4/\hat{a}_r^2] = \hat{a}_\theta^2[C_1 + C_3/\hat{a}_\theta^4] + \hat{a}_r^2[C_2 + C_4/\hat{a}_r^4]$$
$$\approx \hat{a}_\theta^2[\alpha + \alpha] + \hat{a}_r^2[\alpha + \alpha]$$
$$= 2\alpha[\hat{a}_\theta^2 + \hat{a}_r^2] \quad (47)$$

Using α in Eq. (44) gives

$$\text{Group 2} \approx [\hat{a}_\theta^2 + \hat{a}_r^2][k^{1/2}/\beta\mu^2 a_N^2] \quad (48)$$

This form in Eq. (48) is equivalent to the physical cost functional integrand L_1, given in Eq. (22).

VI. Guidance and the Modified Functional J_2

The method of obtaining guidance corrections for perturbations or disturbances utilizes analytic coefficients and thus eliminates many of the aforementioned numerical problems.

The modified cost J_2 has been shown to be close to the physical cost J_1. Thus, the second-variation method in the calculus of variations applied to J_1 or J_2 should yield similar results because the system dynamics in both cases are the same. Since the solutions to the minimum of the modified cost are known analytically in Eqs. (11, 16, 36, and 37), great computational advantages accrue if J_2 is used.

The interested reader can pursue the heuristic mathematical analysis in Ref. 9. The analysis indicates the equivalence of the second-variation method to the variation of the multipliers in the Euler-Lagrange equations necessary for a minimum.[4,5] Assume that, for the problem with disturbances, the new optimal solutions are

$$\lambda = \hat{\lambda} + \delta\lambda,\quad x = \hat{x} + \delta x,\quad m = \hat{m} + \delta m \quad (49)$$

Substitution of Eq. (49) into the Euler-Lagrange equations for the first variation of the disturbed system yields the perturbation equations[9]

$$-d\delta\lambda/dk = [\partial^2\hat{H}_2/\partial x^2]\delta x + [\partial^2\hat{H}_2/\partial m\partial x]\delta m + \delta\lambda^T[\partial\hat{f}/\partial x] \quad (50)$$

$$d\delta x/dk = (\partial\hat{f}/\partial x)\delta x + (\partial\hat{f}/\partial m)\delta m;\ \delta x(k_0) = \delta x_0 \quad (51)$$

$$\delta\lambda(k_f) = (\partial^2\hat{G}/\partial x^2)\delta x(k_f) \quad (52)$$

$$0 = (\partial^2\hat{H}_2/\partial m^2)\delta m + (\partial^2\hat{H}_2/\partial x\partial m)\delta x + \delta\lambda^T\partial\hat{f}/\partial m \quad (53)$$

All coefficients of δx, δm, and $\delta\lambda$ in Eqs. (50–53) are known analytically as functions of k along the reference trajectory (ˆ).

The order of the system of differential equations in Eqs. (51) and (52) can be cut in half by noting that the equations for $\delta\theta$, δt, $\delta\lambda_1$, and $\delta\lambda_4$ can be uncoupled from the remainder of the system. First, looking at the adjoint equations of Eq. (50), the indicated partial derivatives are performed using the definition of H_2 in Eq. (24) along with the integrand L_2 in Eq. (44), and the system dynamics f_i of Eq. (2). Then the reference trajectory of Eqs. (11, 16, 36, and 37) with $\hat{\lambda}_2 = \hat{\lambda}_4 = 0$ is substituted to yield

$$d\delta\lambda_1/dk = 0 \quad (54)$$

$$\frac{d\delta\lambda_2}{dk} = \left\{\frac{3\mu}{\beta k^2}\right\}\delta\lambda_1 + \left[\frac{1}{k}\right]\delta\lambda_2 + \left\{\frac{\mu^2}{\beta k^{5/2}}\left(1 - \eta\beta - \frac{3\beta^2}{2}\right)\right\}\delta\lambda_3 + \left\{\frac{1}{\beta\mu k^{1/2}}\right\}\delta\lambda_4 - \left\{\frac{2\hat{A}_2\mu}{k\mu_m}(\beta^2 + \eta\beta) + \frac{18\hat{A}_3\mu^4}{k^4}\left(1 - \frac{2}{3}\eta\beta - \frac{5}{6}\beta^2\right)\right\}\delta r + \left\{\frac{\hat{A}_2 k^{1/2}}{\beta\mu\mu_m}(\beta^2 + \eta\beta)\right\}\delta v_r + \left\{\frac{2\hat{A}_3}{k}\right\}\delta a_r - \left\{\frac{2\hat{A}_2 k^2}{\beta\mu^3\mu_m}(\beta^2 + \eta\beta) + \frac{4\hat{A}_3}{\beta k}\left(1 - \eta\beta - \frac{3}{2}\beta^2\right)\right\}\delta a_\theta \quad (55)$$

$$\frac{d\delta\lambda_3}{dk} = -\left[\frac{k^{1/2}}{\beta\mu^2}\right]\delta\lambda_2 + \left[\frac{\hat{A}_2 k^{1/2}}{\beta\mu\mu_m}(\beta^2 + \eta\beta)\right]\delta r - \left[\frac{\hat{A}_3}{k} + \frac{3\hat{A}_2 k^2}{\beta^2\mu^3\mu_m}(\beta^2 + \eta\beta)\right]\delta v_r + \left[\frac{2\hat{A}_2 k^{7/2}}{\beta^2\mu^5\mu_m}(\beta^2 + \eta\beta)\right]\delta a_\theta \quad (56)$$

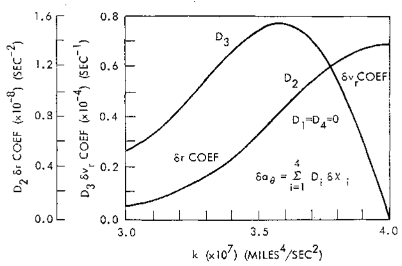

Fig. 3 Guidance coefficients for δa_θ.

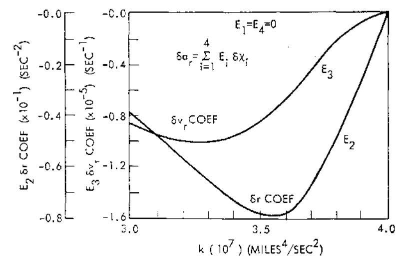

Fig. 4 Guidance coefficients for δa_r.

$$d\delta\lambda_4/dk = 0 \tag{57}$$

where the quantities $\hat{A}_2 = A_2/r_N^2$, and $\hat{A}_3 = A_3/v_N^2$.

The boundary conditions from Eqs. (52) and (19) are

$$\delta\lambda_1(k_f) = 2A_1[\delta\theta(k_f) - \dot{\theta}_T\delta t(k_f)]/\theta_n^2 \tag{58}$$

$$\delta\lambda_2(k_f) = 2A_2\delta r(k_f)/r_n^2 \tag{59}$$

$$\delta\lambda_3(k_f) = 2A_3\delta v_r(k_f)/v_n^2 \tag{60}$$

$$\delta\lambda_4(k_f) = -2A_1\dot{\theta}_T[\delta\theta(k_f) - \dot{\theta}_T\delta t(k_f)]/\theta_n^2 \tag{61}$$

Look first at the solutions for $\delta\lambda_1$ and $\delta\lambda_4$. Equations (54) and (57) show that they are constant. Combining these solutions with the boundary conditions of Eqs. (57) and (61) yields

$$\delta\lambda_1 = \text{const} = 2A_1[\delta\theta(k_f) - \dot{\theta}_T\delta t(k_f)]/\theta_N^2 \tag{62}$$

$$\delta\lambda_4 = \text{const} = -2A_1\dot{\theta}_T[\delta\theta(k_f) - \dot{\theta}_T\delta t(k_f)]/\theta_N^2 \tag{63}$$

Looking at the physical problem, it is noted that the value of $\delta\theta(k_f)$ is the amount of additional angular distance, beyond the nominal terminal angular $\theta(k_f)$, traveled by the low-thrust vehicle. Similarly, $\dot{\theta}_T\delta t(k_f)$ is the amount of additional angular distance, beyond the nominal terminal angular distance $[\hat{t}(k_f) - t_0]\dot{\theta}_T$ traveled by the unthrusted target orbiter in its circular orbit. When no disturbances are present, the two vehicles have traveled equal terminal angular distances in space, i.e., $\hat{\theta}(k_f) = [t(k_f) - t_0]\dot{\theta}_T$. Thus, the rendezvous condition, where disturbances are present becomes

$$\delta\theta(k_f) = \dot{\theta}_T\delta t(k_f) \tag{64}$$

Assuming the rendezvous condition is met yields the final solutions for the perturbed multipliers $\delta\lambda_1$ and $\delta\lambda_4$

$$\delta\lambda_1(k) = 0,\ \delta\lambda_4(k) = 0 \tag{65}$$

The differential equations for the state variatons of δr and δv_r are easily shown to be uncoupled from the variations $\delta\theta$ and δt. Thus, taking the indicated partial derivatives in Eq. (51) and evaluating them along the reference trajectory ($\hat{\ }$) yields

$$d\delta r/dk = -[1/k]\delta r + [k^{1/2}/\beta\mu^2]\delta v_r - [2k^2/\beta\mu^4]\delta a_\theta \tag{66}$$

$$\frac{d\delta v_r}{dk} = -\left[\frac{\mu^2}{\beta k^{5/2}}\left(1 - \eta\beta - \frac{3\beta^2}{2}\right)\right]\delta r + \left[\frac{k^{1/2}}{\mu^2}\right]\delta a_\theta + \left[\frac{k^{1/2}}{\beta\mu^2}\right]\delta a_r \tag{67}$$

The equations for the perturbed controls, in Eq. (53), after taking the indicated partial derivatives, and evaluating them along the reference trajectory ($\hat{\ }$) become

$$0 = \left[8\alpha + \frac{4\hat{A}_3k^5}{\mu^4} - \frac{4\hat{A}_2k^5}{\beta^2\mu^7\mu_m}(\beta^2 + \eta\beta)\right]\delta a_\theta - \left[\frac{4\hat{A}_3k^2}{\beta\mu^4}\right]\delta a_r - \left[\frac{2k^2}{\beta\mu^4}\right]\delta\lambda_2 + \left[\frac{k^{1/2}}{\mu^2}\right]\delta\lambda_3 - \left[\frac{2\hat{A}_2k^{7/2}}{\beta^2\mu^5\mu_m}\right](\beta^2 + \eta\beta)\delta v_r + \left[\frac{2\hat{A}_2k^2}{\beta\mu^3\mu_m}(\beta^2 + \eta\beta) + \frac{4\hat{A}_3}{\beta k}\left(1 - \eta\beta - \frac{3\beta^2}{2}\right)\right]\delta r \tag{68}$$

$$0 = [8\alpha + 12\hat{A}_3k^2/\eta\beta\mu^4]\delta a_r - [4\hat{A}_3k^2/\beta\mu^4]\delta a_\theta - [2\hat{A}_3/k]\delta r + [k^{1/2}/\beta\mu^2]\delta\lambda_3 \tag{69}$$

Note that solutions for the controls in Eqs. (68) and (69) require inversions of a matrix, i.e., $[\partial^2\hat{H}/\partial m^2]^{-1}$. Since the reference trajectory is known analytically, the matrix inversion may be performed analytically as:

$$\begin{bmatrix}\dfrac{\partial^2\hat{H}}{\partial m^2}\end{bmatrix}^{-1} = \begin{bmatrix}\text{I} & -b\\ -b & \text{II}\end{bmatrix}^{-1} = \begin{bmatrix}\text{II/DET} & b/\text{DET}\\ b/\text{DET} & \text{I/DET}\end{bmatrix} \tag{70}$$

where

$$\begin{aligned}\text{I} &= [8\alpha + 4\hat{A}_3k^2/\mu^4 - 4\hat{A}_2k^5/\beta^2\mu^7\mu_m]\\ \text{II} &= [8\alpha + 12\hat{A}_3k^2/\eta\beta\mu^4],\ b = 4\hat{A}_3k^2/\beta\mu^4\\ \text{DET} &= [\text{I}\cdot\text{II} - b^2] \neq 0\end{aligned} \tag{71}$$

Thus, no numerical matrix inversions are required.

VII. Numerical Problem

The simultaneous solution of Eqs. (55, 56, and 66–69) yields the perturbed controls for rendezvous. These solutions may be obtained in feedback form[9] by utilizing a Ricatti transformation

$$\delta\lambda(k) = F(k)\delta x(k) \tag{72}$$

The optimal correction control δm becomes the feedback control:

$$\delta m(k) = -[\partial^2\hat{H}_2/\partial m^2]^{-1}\{\partial^2\hat{H}_2/\partial x\partial m + \partial\hat{f}^T/\partial m\, F\}\delta x(k) \tag{73}$$

The results for the system of equations with parameters as in Eq. (45) are given in Figs. 3 and 4.

VIII. Conclusions

1) Small onboard computers cannot handle the numerical problems associated with the fuel optimal rendezvous expressed in the cost functional J_1.

2) The transformation of the equations of motion into the k domain yields new closed form programed controls. Since they are analytic, there is no need to iterate or store their values along the trajectory. This transformation also yields the great computational advantage of changing the open terminal time problem into one with a fixed final value of the independent variable k.

3) The paper determines a modified cost to be used instead of the original cost J_1. The change ΔJ_1 is small. The minimization of J_2 yields the controls in conclusion 2 as pseudo-optimal.

4) The linear guidance law from the second variation of the modified cost functional J_2 is given in Eq. (73) and Figs. 3 and 4. All coefficients are found analytically along the reference trajectory. The vehicle follows a neighboring pseudo-optimal trajectory in the k domain to rendezvous when disturbances exist. Fuel losses are also kept low. Thus, the analytic solutions eliminate the numerical storage problems of the coefficients. Additionally, the necessity of numerical matrix inversions are also eliminated.

5) Figure 2 indicates very good comparisons with previous work[2] (fuel minimization for specified energy of escape). Even for large radial distance r, the control angle $\psi = \tan^{-1}[\hat{a}_r/\hat{a}_\theta] = \tan^{-1}[\eta]$ is very close to the direction of optimal increase of energy.

6) Good performance of the guidance scheme utilized in the paper depends upon J_1 and J_2 being close together. Reference 9 indicates that the guidance equations applied to J_1 or J_2 have negligible differences. A future paper by the authors will discuss the problem of decreasing thrust magnitude with altitude and will develop a compensating control scheme.

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Optimum Estimation and Coordinate Conversion for Radio Navigation

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Radio navigation aids such as OMEGA and LORAN can provide many hyperbolic lines of position (LOP's) that, in general, do not have a common point of intersection. This paper describes a technique that uses the extra LOP's which are linearly independent to provide the most probable receiver location in geodetic coordinates. The approximate nonlinear algebraic equations that maximize the a posteriori density function are given and a two-dimensional Newton-Raphson scheme presented for their solution. The covariance of the geodetic estimation error is shown to be the inverse of the Jacobian matrix associated with the Newton-Raphson scheme for small measurement noise. This matrix is used to develop accuracy contour plots for the four OMEGA transmitting sites that are presently on the air. An improvement in OMEGA accuracy of about one-half mile can be expected when this four station technique is used to replace a conventional three station coordinate conversion scheme.

Nomenclature

ϕ,λ = receiver latitude and longitude, respectively
$\hat{\phi},\hat{\lambda}$ = estimates of ϕ and λ, respectively
t_i = time-of-arrival of signal from the ith transmitter
d_i = geodetic distance between receiver and ith transmitter
c = speed of light
n = number of transmitters
$\mathbf{D}$ = $(n - 1) \times 1$ vector of distance differences
$\mathbf{T}$ = $(n - 1) \times 1$ vector of time differences
$\boldsymbol{\varepsilon}$ = $(n - 1) \times 1$ vector of time difference errors
$\mathbf{S}$ = $(n - 1) \times 1$ vector equal to $\mathbf{T} - (1/c)\mathbf{D}$
$\boldsymbol{\Phi}$ = $(n - 1) \times (n - 1)$ covariance matrix of time difference errors
σ_i^2 = variance of time of arrival measurement from the ith transmitter
θ_i = azimuth angle of transmitter i measured at receiver
R = earth radius
$C_n(2)$ = number of combinations of n things taken two at a time

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Introduction

THE expansion of the OMEGA hyperbolic navigation system to include eight ground based radio transmitters poses questions of the following type: 1) if redundant lines of position (LOP's) are available with many intersecting points, how should the information be combined to provide the most probable position of the receiver? 2) neglecting lane ambiguity, should all the LOP's or even possibly all the intersection points of the LOP's be used in calculating the most probable receiver location? 3) is it necessary to coordinate convert all the LOP information to geodetic coordinates before statistically averaging or can the averaging and coordinate conversion be done simultaneously?

This paper answers these questions by describing a technique that uses the minimum LOP information necessary to obtain a most probable estimate of receiver location. The estimate is obtained in the desired geodetic coordinate system directly from hyperbolic time difference information, i.e.,